Answer of Mid Term for Mathematic 2B

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(1) For Legendre polynomial $\boldsymbol{P_m}$ Prove that $\displaystyle\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx = \frac{2m}{4m^2-1}$

(2) Use generating function for Legendre polynomial to show that

$$P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} \quad \text{and} \quad P_{2m+1}(0) = 0$$

(3) For the Bessel Function $J_m(x)$ show that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$.

(4) Use the relation $J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$ to find $J_{3/2}(x)$ and $J_{-3/2}(x)$.

<u>Answer (1)</u>

$$\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx = \int_{-1}^{1}\left[\frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x)\right]P_{m-1}(x)dx$$

$$= \frac{m+1}{2m+1}\int_{-1}^{1} P_{m+1}(x)P_{m-1}(x)dx + \frac{m}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx$$

$$= 0 + \frac{m}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx = \frac{2m}{(2m+1)(2m-1)} = \frac{2m}{(4m^2-1)}$$

<u>Answer (2)</u>

Expanding the left-hand side by the binomial theorem gives us

$$\frac{1}{\sqrt{(1+t^2)}} = [1+t^2]^{-1/2}$$

$$= 1 + (-\tfrac{1}{2})t^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(t^2)^2 + \ldots + \frac{(-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{2m-1}{2})}{m!}(t^2)^m + \ldots$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1.3.5\ldots(2m-1)}{2^m\, m!} t^{2m}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1.2.3.4.5...(2m-1)2\text{m}}{2^m (m!).2.4.6...2\text{m}} t^{2m}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1.2.3.4.5...(2m-1)2\text{m}}{2^m (m!)2^m.(1.2.3...\text{m})} t^{2m}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^{2m} (m!)^2} t^{2m}$$

Then $= \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^{2m} (m!)^2} t^{2m} = \sum_{m=0}^{\infty} t^m P_m(0)$

Equating coefficients of corresponding powers of t on both sides

gives $P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m} (m!)^2}$ and $P_{2m+1}(0) = 0$

Answer (3)

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(r+\frac{3}{2})} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

$$= \sqrt{\frac{2}{x}} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{2^{2r+1}(r!)\Gamma(r+\frac{3}{2})}$$

$$= \sqrt{\frac{2}{x}} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{2^{2r+1}(r!)(\text{r}+\frac{1}{2})\Gamma(r+\frac{1}{2})}$$

Since $\Gamma(r+\frac{1}{2}) = \frac{(2r)!\sqrt{\pi}}{2^{2r}\, r!}$ Substitute in $J_{1/2}(x)$ (8)

$$=\sqrt{\frac{2}{x}}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}.2^r.r!}{2^{2r+1}(r!)(r+\frac{1}{2})(2r)!\sqrt{\pi}}$$

$$=\sqrt{\frac{2}{x\pi}}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)(2r)!}$$

$$=\sqrt{\frac{2}{\pi x}}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}=\sqrt{\frac{2}{\pi x}}\sin x \quad \blacksquare$$

Answer (4)

Since $J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$

At $n=\frac{1}{2}$ we have $J_{\frac{-1}{2}}(x)+J_{\frac{3}{2}}(x)=\frac{1}{x}J_{\frac{1}{2}}(x)$

Then $J_{\frac{3}{2}}(x)=\frac{1}{x}J_{\frac{1}{2}}(x)-J_{\frac{-1}{2}}(x)=\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x=\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x}-\cos x\right]$

At $n=\frac{-1}{2}$ we have $J_{\frac{-3}{2}}(x)+J_{\frac{1}{2}}(x)=\frac{1}{x}J_{\frac{-1}{2}}(x)$

Then

$$J_{\frac{-3}{2}}(x)=\frac{1}{x}J_{\frac{-1}{2}}(x)-J_{\frac{1}{2}}(x)=\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x-\sqrt{\frac{2}{\pi x}}\sin x=-\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x}-\sin x\right]$$

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